Bifurcation Analysis of Aircraft Pitching Motions About Large Mean Angles of Attack

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Bifurcation theory is used to analyze the nonlinear dynamic stability characteristics of an aircraft subject to single-degree-of-freedom pitching-motion perturbations about a large mean angle of attack. The requisite aerodynamic information in the equations of motion can be represented in a form equivalent to the response to finite-amplitude pitching oscillations about the mean angle of attack. It is shown how this information can be deduced from the case of infinitesimal-amplitude oscillations. The bifurcation theory analysis reveals that when the mean angle of attack is increased beyond a critical value at which the aerodynamic damping vanishes, new solutions representing finite-amplitude periodic motions bifurcate from the previously stable steady motion. The sign of a simple criterion, cast in terms of aerodynamic properties, determines whether the bifurcating solutions are stable (supercritical) or unstable (subcritical). For flat-plate airfoils flying at supersonic/hypersonic speed, the bifurcation is subcritical, implying either that exchanges of stability between steady and periodic motion are accompanied by hysteresis phenomena, or that potentially large aperiodic departures from steady motion may develop.

Nomenclature

C_m	= pitching-moment coefficient
D	= damping-in-pitch derivative
h	= distance of center of gravity from wing leading edge, made dimensionless with respect to reference length ℓ
I	= moment of inertia
ℓ	= reference length
M	= instantaneous pitching moment
M_∞	= freestream Mach number
p,p_∞	= local and freestream pressure, respectively
S	= stiffness derivative
$\bar{S}$	= reference area
t	= time
v	= velocity vector
V_∞	= freestream velocity
κ	= dimensionless parameter $\rho_\infty \bar{S}\ell^3/2I$
γ	= ratio of specific heats
ξ	= instantaneous inclination of angle of attack from mean angle of attack, $\sigma - \sigma_m$
ρ,ρ_∞	= local, freestream density
σ	= instantaneous angle of attack, inclination of aircraft longitudinal axis from direction of aircraft velocity vector
σ_{cr}	= critical angle of attack, value of σ at which $D=0$
σ_m	= mean angle of attack
τ	= dimensionless time, $V_\infty t/\ell$

I. Introduction

PROBLEMS of aerodynamic stability of aircraft flying at small angles of attack have been studied extensively. With increasing angles of attack the problems become more complicated and typically involve nonlinear phenomena such as coupling between modes, amplitude and frequency effects, and hysteresis. The need for investigating stability characteristics at high angles of attack was clearly demonstrated by Orlik-Rückemann[1] in his survey paper which largely deals with experiments.

On the theoretical side, the greater part of an extensive body of work is based on the linearized theory, in which the unsteady flow is regarded as a small perturbation of some known (possibly nonlinear in, say, the angle of attack) steady flow that prevails under certain flight conditions. The question of the validity and limitations of such a linearized perturbation theory is of fundamental importance and yet has been investigated only rarely. One may argue that, in principle, it should be possible to advance to higher and higher angles of attack, σ, by a series of linear perturbations, since the solution at each step should include a steady-state part which, when added to the previous steady-state solution, would provide the starting point for the next perturbation. This may well be true provided that at each step the steady motion is stable both statically and dynamically, and that the actual disturbances, for example, the amplitude of oscillation, remain small. However, the linear perturbation procedure must eventually cease to be valid when the angle of attack exceeds a certain critical value σ_{cr} at which the steady motion is no longer stable.

In this paper we investigate the stability characteristics of an aircraft trimmed to a mean angle of attack σ_m near σ_{cr}, at which the steady motion becomes unstable. Padfield[2] studied a similar problem, using the method of multiple scales, which is valid only for weakly nonlinear oscillations. We shall study the problem by means of bifurcation theory. This will allow us to draw on recent mathematical developments (e.g., Ref. 3) that are particularly well suited to investigating fundamental questions in both linear and nonlinear stability. A numerical scheme based on bifurcation theory was proposed earlier by Mehra and Carroll[4] for analyzing aircraft dynamic stability in a rather general framework. More recently, Guicheteau[5] has demonstrated the considerable potential of bifurcation theory in flight dynamics studies, particularly toward establishing a method for the design of flight control systems insuring

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protection against loss of control. On the other hand, while acknowledging the importance of the aerodynamic model in determining the aircraft's stability characteristics, neither of these works contains an adequate assessment of the model's requirements. The treatment of unsteady flow effects, in particular, receives no attention. In contrast, we shall focus on just this aspect of the problem at the expense of narrowing the scope of the motion analysis. Restricting the motion to a single-degree-of-freedom pitching oscillation will enable us to analyze a motion for which complete aerodynamic information is available, for certain aerodynamic shapes, in the form of exact solutions of the inviscid supersonic/hypersonic unsteady flow theory.[6-10] In this way it will be possible to establish a form revealing a precise analytical relationship between the basic aerodynamic coefficients and the characteristics of the aircraft's motion.

II. Mathematical Formulation

A. Equations of Motion

To fix ideas in a concrete example, we restrict attention to the single-degree-of-freedom pitching oscillations of an aircraft about a fixed trim angle. The motion, of course, can be simulated in an oscillations-in-pitch experiment in the wind tunnel, so that focusing on this motion ultimately will facilitate an experimental confirmation of the theoretical results.

Before time zero, let the aircraft be in level steady flight with its longitudinal axis inclined from the horizontal velocity vector by the fixed trim angle of attack σ_m. At time zero the aircraft is perturbed from its trim position, but during the subsequent motion the center of gravity continues to follow a rectilinear path at constant velocity V_∞. The instantaneous angle of attack $\sigma(t)$ is again measured relative to the horizontal velocity vector, while the inclination of $\sigma(t)$ from the fixed trim angle σ_m, $\sigma(t)-\sigma_m$, will be designated $\xi(t)$. Written as a pair of first-order differential equations, the equations of motion are

$$\frac{\mathrm{d}}{\mathrm{d}t}(\sigma-\sigma_m)=\dot{\xi} \tag{1a}$$

$$I\frac{\mathrm{d}\dot{\xi}}{\mathrm{d}t}=M(t) \tag{1b}$$

where I is the moment of inertia and $M(t)$ the instantaneous aerodynamic pitching moment, both referred to the center of gravity. We assume that the moment required to trim the aircraft at σ_m has been accounted for, so that $M(t)$ is a measure of the perturbation moment only.

Now we consider the equations governing the flow around the aircraft, assuming for simplicity that it is permissible to neglect viscous effects. The inviscid unsteady flowfield equations are

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\boldsymbol{v})=0 \tag{2a}$$

$$\frac{\partial\boldsymbol{v}}{\partial t}+\boldsymbol{v}\cdot\nabla\boldsymbol{v}+\frac{\nabla p}{\rho}=0 \tag{2b}$$

$$\frac{\partial}{\partial t}\frac{p}{\rho\gamma}+\boldsymbol{v}\cdot\nabla\frac{p}{\rho\gamma}=0 \tag{2c}$$

where p, ρ, and $\boldsymbol{v}$ are the pressure, density, and velocity vector, respectively, and γ the ratio of specific heats of the medium. Denote the position vector originating from the center of gravity by $\boldsymbol{r}$ and let the equation of the body surface be $B(\boldsymbol{r},t)=0$, which, of course, is dependent on the instantaneous displacement angle $\xi(t)$. The tangency condition at the body surface is then

$$\frac{\partial B}{\partial t}+\boldsymbol{v}\cdot\nabla B=0 \quad \text{on} \quad B=0 \tag{3}$$

which is also dependent on $\xi(t)$. The far-field boundary conditions (relative to a coordinate system fixed in the aircraft) are

$$p=p_\infty, \qquad \rho=\rho_\infty \qquad \boldsymbol{v}=\boldsymbol{V}_\infty \tag{4}$$

for subsonic flight, or the well-known shock jump conditions for supersonic flight.

It is clear from the above that the pitching motion of the aircraft $\xi(t)$ is coupled to the unsteady flowfield (e.g., pressure, p) through Eq. (3), and, more directly, through $M(t)$ in Eq. (1), which is determined from the instantaneous surface pressure through

$$M(t)=\int\!\!\int_{S:B=0}\boldsymbol{r}\times p\frac{\nabla B}{|\nabla B|}\,\mathrm{d}S \tag{5}$$

The principal difficulty, however, is that the instantaneous surface pressure in Eq. (5) depends not only on the current state of the flowfield but also on its prior states. This means that past values of ξ figure in the determination of the current state of the flowfield. Thus, unless approximations are introduced, a determination of ξ starting from given initial conditions requires the simultaneous solution of the coupled equations (1-5).

Although the simultaneous solution of the coupled equations in principle represents an exact approach to the problem of determining time histories of maneuvers from given initial conditions, it is inevitably a difficult and costly approach (see the discussion of Chyu and Schiff).[11] Approximate approaches leading to simpler, less costly computations are a practical necessity. To date, these have had in common invoking the assumption of a slowly varying motion of known form (e.g., a harmonic motion) whose aerodynamic force and moment response at t can be calculated. This stratagem in effect uncouples the flowfield equations from the equations of motion. In a series of papers (see in particular Ref. 12), Tobak and Schiff have shown how this stratagem can be rationalized to create mathematical models of the aerodynamic response at various levels of approximation of the dependence on the past motion. The characteristic features of the second level, which are all that will concern us here, are evident from the form of the instantaneous perturbation pitching moment

$$M(t)=\tfrac{1}{2}\rho_\infty V_\infty^2\bar{S}\ell[C_m(t)-C_m(\infty;\sigma_m)] \tag{6}$$

where[12]

$$C_m(t)=C_m\big(\infty;\sigma(t)\big)-\int_0^t F\Big(\frac{V_\infty}{\ell}(t-t_I);\sigma(t_I)\Big)\frac{\mathrm{d}\sigma}{\mathrm{d}t_I}\mathrm{d}t_I \tag{7}$$

Here ρ_∞ and V_∞ are the freestream density and velocity, respectively, and $\bar{S}$ and ℓ are the reference area and length. The function $C_m\big(\infty;\sigma(t)\big)$ is the pitching-moment coefficient that would be measured in steady flow with the angle of attack fixed at the instantaneous angle of attack $\sigma(t)$. Similarly, $C_m(\infty;\sigma_m)$ is the steady-state pitching-moment coefficient corresponding to the fixed trim angle of attack σ_m. The integral term is a representation of the transient nature of the pitching-moment response to changes in angle of attack. The response at time t depends principally on the motion in the recent past relative to t, which, when the motion is slowly varying, can be characterized with sufficient accuracy by the instantaneous angle of attack $\sigma(t)$ and the instantaneous rate of change of angle of attack $\dot{\sigma}(t)$.

On the assumption that terms of $0(\dot{\sigma}^2,\ddot{\sigma})$ and higher can be neglected for slowly varying motions, the integral term in Eq. (7) can be simplified to yield a form correct to within a linear dependence on $\dot{\sigma}(t)$. Equation (7) takes the form

$$C_m(t)=C_m\big(\infty;\sigma(t)\big)+\dot{\sigma}(t)\frac{\ell}{V_\infty}C_{m_{\dot{\sigma}}}\big(\sigma(t)\big) \tag{8}$$

where

$$C_{m_{\dot\sigma}}\left((\sigma(t)\right) = -\int_0^{\tau_a} F\left(u;\sigma(t)\right)\mathrm{d}u, \qquad \tau_a = \frac{V_\infty t_a}{\ell} \tag{9}$$

The term $C_{m_{\dot\sigma}}(\sigma(t))$ is the damping-in-pitch coefficient that would be measured from infinitesimal pitching oscillations about a mean value of angle of attack equal to $\sigma(t)$. Alternatively, since $C_{m_{\dot\sigma}}$ is assumed to depend only on $\sigma(t)$, within the assumption the response to *any* of a class of sufficiently slowly varying motions which arrives at the same value of σ at time t will yield the same value of $C_{m_{\dot\sigma}}$. In particular, a finite-amplitude harmonic pitching oscillation about the mean angle of attack σ_m can be used as well to obtain $C_{m_{\dot\sigma}}$ as long as the instantaneous inclination $\xi(t)$ is equal to $(\sigma(t)-\sigma_m)$. This, then, is the rationale that allows us to decouple the flowfield equations from the inertial equations of motion and to use results for the aerodynamic terms in the mathematical model from known harmonic motions.

Chyu and Schiff[11] have shown, via numerical computations for the case of an oscillating flap at transonic speed, that use of a form equivalent to that of Eq. (8) for the aerodynamic model yields results in excellent agreement with computational results based on the coupled-equations approach. Moreover, a comparison of the computational costs involved demonstrated the relative economy of the mathematical modeling approach. (Indeed, the computations required in the mathematical modeling approach have the potential of significantly greater economy than was demonstrated in Ref. 11 if, instead of the method used therein, the aerodynamic responses in the modeling approach were calculated from gasdynamic equations *linearized* about instantaneous position. This simplification is consistent with the second level of approximation.)

In the next section we note that modeling on the basis of Eq. (8) is particularly appropriate near the boundary for neutral dynamic stability.

B. Neutral Dynamic Stability Boundary

Let us suppose that there is a mean angle of attack $\sigma_m = \sigma_{cr}$ at which the damping-in-pitch coefficient vanishes. Then, at and beyond this angle of attack, the steady motion of the aircraft no longer will be stable so that, in response to a disturbance, the motion will seek a new stable state which usually will consist of a finite-amplitude periodic oscillation about the mean angle of attack. The changeover from a stable steady motion to a stable periodic motion is called a Hopf bifurcation (see Refs. 3, 13; also Sec. IV). Thus, in the vicinity of σ_{cr}, where the damping-in-pitch coefficient will be near zero, the consequent persistence of oscillatory motions will make it particularly appropriate to assume a periodic past motion about a mean position in calculating the aerodynamic pitching moment from the flowfield equations for use in Eq. (1).

As we have noted, this assumption is consistent with the mathematical modeling approach of Tobak and Schiff[12] at the second level of approximation. It will be convenient to rewrite Eq. (6) in the form

$$M(t) = \tfrac{1}{2}\rho_\infty V_\infty^2 \bar{S}\ell\left[C_m(\xi,\dot\xi,\sigma_m) - C_m(0,0,\sigma_m)\right] \tag{10}$$

The function $C_m(\xi,\dot\xi,\sigma_m)$, equivalent to Eq. (8), is the pitching-moment coefficient resulting from a finite-amplitude periodic pitching motion $\xi(t)$ about the center of gravity, and $C_m(0,0,\sigma_m)$ is its steady-state value at the mean angle of attack σ_m. The latter term, of course, is equal to $C_m(\infty;\sigma_m)$ in Eq. (6). The function $C_m(\xi,\dot\xi,\sigma_m)$ depends on the instantaneous displacement angle, $\xi(t)$, its rate of change, $\dot\xi(t)$, and the mean angle of attack, σ_m. It also depends, of course, on the location, h, of the center of gravity relative, say, to the wing leading edge, the flight Mach number M_∞, the ratio of the specific heats of the medium γ, and the aircraft shape. A great amount of work has been devoted to the [illegible]

pitching oscillations of infinitesimal amplitude. For example, for the cases of a wedge, a flat plate in supersonic/hypersonic flow[7,8] and an airfoil of arbitrary profile in the Newtonian limit,[9] this function is available in exact analytical form for large, as well as small, mean angle of attack σ_m, including the critical angle σ_{cr}. On the other hand, little is known about the pitching-moment coefficient C_m when the amplitude of oscillation is finite, except for the special case of a slowly oscillating wedge.[10] In the next section we shall show how the function C_m for slow pitching oscillations of finite amplitude may be obtained from its behavior in the limit of oscillations of infinitesimal amplitude.

C. Pitching-Moment Function for Slow Pitching Oscillations of Finite Amplitude

Consider a uniform flow V_∞ past an aircraft that is undergoing a slow pitching oscillation of finite amplitude about a mean angle of attack σ_m. The instantaneous angular displacement from the mean position is measured by $\xi(t)$ so that $|\xi|_{\max}$ is the finite amplitude of the oscillation. It is required to calculate the unsteady pressure field and hence the form of the pitching moment $M(t)$ in Eq. (1). Let the cylindrical coordinates (r,ϕ,z) be such that the z axis coincides with the lateral axis through the center of gravity. Let the equation of the body surface at the mean angle of attack σ_m be $\phi = \sigma_m + A(r,z)$. Then its equation at time t is

$$B(r,\phi,z,t) = A(r,z) + \sigma_m + \xi(t) - \phi = 0 \tag{11}$$

With velocity vector $v = (v_r, v_\phi, v_z)$, the boundary condition Eq. (3) becomes

$$v_\phi = r\left[\dot\xi + v_r\frac{\partial A}{\partial r} + v_z\frac{\partial A}{\partial z}\right] \tag{12}$$

at $\phi = A(r,z) + \sigma_m + \xi(t)$. Equations (2), (4), and (12) complete the formulation for calculating the flowfield in terms of $\xi(t)$. Now, for a long-established periodic motion of the aircraft, the aerodynamic response is also periodic. Furthermore, invoking the assumption of *slow* oscillations allows us to neglect terms of $0(\dot\xi^2,\ddot\xi)$ and higher and to write

$$p = p_0 + \dot\xi p_1, \quad \rho = \rho_0 + \dot\xi\rho_1 \quad v = v_0 + \dot\xi v_1 \tag{13}$$

where $(\ )_0$ and $(\ )_1$ are independent of $\dot\xi(t)$, but may depend on time through the function $\xi(t)$. Since the zeroth-order quantities $(\ )_0$ represent the flowfield when $\dot\xi = 0$, such a flow must be quasisteady; that is, the time variable t can only appear implicitly through the instantaneous displacement angle $\xi(t)$. Hence,

$$p_0 = p_0(\sigma_m + \xi(t), r, \phi, z), \quad \text{etc.} \tag{14}$$

To derive the mathematical problem for the first-order quantities $(\ )_1$, we substitute Eq. (13) into Eqs. (2), (4), and (12), use Eq. (14), and neglect terms of $0(\dot\xi^2,\ddot\xi)$. Thus,

$$\nabla\cdot(\rho_0 v_1 + \rho_1 v_0) = -\frac{\partial\rho_0}{\partial\xi} \tag{15a}$$

$$v_0\cdot\nabla v_1 + v_1\cdot\nabla v_0 + \frac{1}{\rho_0}\left[\nabla p_1 - \frac{\rho_1}{\rho_0}\nabla p_0\right] = -\frac{\partial v_0}{\partial\xi} \tag{15b}$$

$$v_0\cdot\nabla\left(\frac{p_1 - \dfrac{\gamma p_0}{\rho_0}\rho_1}{\rho_0\gamma}\right) + v_1\cdot\nabla\left(\frac{p_0}{\rho_0^{\gamma}}\right) = -\frac{\partial}{\partial\xi}\left(\frac{p_0}{\rho_0^{\gamma}}\right) \tag{15c}$$

$$v_1 = p_1 = \rho_1 = 0 \quad \text{as} \quad |r| \to \infty \tag{15d}$$

$$v_{1_\phi} = r\left[1 + v_{1_r}\frac{\partial A}{\partial r} + v_{1_z}\frac{\partial A}{\partial z}\right]$$

$$\text{at} \quad \phi = A(r,z) + \sigma_m + \xi(t) \tag{15e}$$

Since the time variable t appears in Eqs. (15) only as a parameter through $\xi(t)$, the solution to Eqs. (15) must be of the form

$$p_1 = p_1(\sigma_m + \xi(t), r, \phi, z), \quad \text{etc.} \tag{16}$$

The forms of p_0 and p_1 in Eqs. (14) and (16) determine the form of the pressure p in Eq. (13) which in turn, after the integrations indicated in Eq. (5), determines the form of the pitching-moment coefficient. Thus,

$$C_m(\xi,\dot{\xi},\sigma_m) = f(\sigma_m+\xi) + \frac{\dot{\xi}\ell}{V_\infty} g(\sigma_m+\xi) \tag{17a}$$

In the case of oscillations of infinitesimal amplitude, Eq. (17a) reduces to

$$C_m = f(\sigma_m) + \xi f'(\sigma_m) + \frac{\dot{\xi}\ell}{V_\infty} g(\sigma_m) \tag{17b}$$

It is now clear that the functions $f'(\sigma_m)$ and $g(\sigma_m)$ are related to the stiffness derivative $S(\sigma_m)$ and the damping-in-pitch derivative $D(\sigma_m)$ at an angle of attack σ_m, as defined in classical aerodynamics, by

$$f'(\sigma_m) = -S(\sigma_m), \qquad g(\sigma_m) = -D(\sigma_m) \tag{18}$$

Comparing Eqs. (17a) and (17b), we conclude that knowing the stiffness and damping derivatives from the results of calculations for oscillations of infinitesimal amplitude enables one to obtain immediately the pitching moment coefficient C_m for the corresponding finite-amplitude case. This general conclusion is supported by the form of the exact solution for C_m for the wedge oscillating at large amplitude given in Ref. 10. [To correct a misprint in Ref. 10, a term $hA_4[\cos(\theta-\theta_a)-1]$ should be added to the right-hand side of Eq. (12b) in Ref. 10.]

Finally, noting that $\sigma(t) = \sigma_m + \xi(t)$, we see that both the forms and interpretations of the respective terms in Eqs. (8) and (17a) are equivalent. This confirms that the form of our results for slow oscillations of finite amplitude is consistent with that of the mathematical modeling approach of Tobak and Schiff[12] at their second level of approximation.

Having determined an appropriate form of $M(t)$ for use in the inertial equations of motion, Eqs. (1), let us now rewrite the equations introducing Eq. (17a) along with dimensionless time (i.e., characteristic lengths of travel) $\tau = V_\infty t/\ell$. [Note that we shall retain the symbol $(\dot{\ })$ to designate a time derivative. Henceforth, however, the derivative will be with respect to *dimensionless* time: $(\dot{\ }) \equiv \mathrm{d}/\mathrm{d}\tau$]. Let

$$F(\xi,\dot{\xi},\sigma_m) = \frac{M(\tau)}{I(V_\infty/\ell)^2} = \kappa[C_m(\xi,\dot{\xi},\sigma_m) - C_m(0,0,\sigma_m)]$$

$$= \kappa[f(\sigma_m+\xi) - f(\sigma_m) + \dot{\xi} g(\sigma_m+\xi)] \tag{19}$$

with

$$\kappa = \frac{\rho_\infty \bar{S}\ell^3}{2I}, \qquad \dot{\xi} = \frac{\mathrm{d}\xi}{\mathrm{d}\tau}$$

The inertial equations of motion become

$$\frac{\mathrm{d}\xi}{\mathrm{d}\tau} = \dot{\xi} \tag{20a}$$

$$\frac{\mathrm{d}\dot{\xi}}{\mathrm{d}\tau} = F(\xi,\dot{\xi},\sigma_m) \tag{20b}$$

An expansion of $F(\xi,\dot{\xi},\sigma_m)$ in a Taylor series in ξ and $\dot{\xi}$ and a change of notation $u_1 = \xi$, $u_2 = \dot{\xi}$ yields for Eqs. (20)

$$\dot{u}_i = A_{ij}(\sigma_m)u_j + B_{ijk}(\sigma_m)u_j u_k + C_{ijk\ell}(\sigma_m)u_j u_k u_\ell$$

$$+ O(|u|^4), \qquad i = 1,2 \tag{21}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -\kappa S(\sigma_m) & -\kappa D(\sigma_m) \end{bmatrix} \tag{22a}$$

$$B_{1jk} = 0, \qquad B_{2jk} = \frac{1}{2!}\frac{\partial^2 F}{\partial u_j \partial u_k}\bigg|_{u=0} \tag{22b}$$

$$C_{1jk\ell} = 0, \qquad C_{2jk\ell} = \frac{1}{3!}\frac{\partial^3 F}{\partial u_j \partial u_k \partial u_\ell}\bigg|_{u=0} \tag{22c}$$

[Although Eqs. (20) have been derived on the assumption of slow oscillations (terms in C_m of $O(\dot{\xi}^2, \ddot{\xi})$ omitted), our subsequent bifurcation analysis of Eqs. (20) will hold for general $F(\xi,\dot{\xi},\sigma_m)$, that is, as if no restriction has been placed on the magnitude of $\dot{\xi}$.]

In Eq. (22) the tensors B and C represent the effects of finite amplitude to the second and third order. We note that the following symmetry properties hold:

$$B_{2jk} = B_{2kj} \tag{23a}$$

$$C_{2jk\ell} = C_{2j\ell k} = C_{2kj\ell} = C_{2k\ell j} = C_{2\ell jk} = C_{2\ell kj} \tag{23b}$$

On the basis of Eq. (21), in subsequent sections we shall study the linear and nonlinear stability of the motion.

III. Linear Stability Theory

The stability of the steady motion at an angle of attack σ_m to *infinitesimal* disturbances is determined by the nature of the eigenvalues of the matrix A. They are

$$\lambda_{1,2}(\sigma_m) = \tfrac{1}{2}[-\kappa D(\sigma_m) \pm \sqrt{\kappa^2 D^2(\sigma_m) - 4\kappa S(\sigma_m)}] \tag{24}$$

Case I: $S(\sigma_m) < 0$. In this case $\lambda_1 > 0$, $\lambda_2 < 0$. The steady motion at this angle of attack σ_m is always unstable.

Case II: $S(\sigma_m) > 0$.

Case IIa: $D(\sigma_m) < 0$. In this case $Re(\lambda_1) > 0$ and the steady motion at σ_m is unstable.

Case IIb: $D(\sigma_m) > 0$. In this case $Re(\lambda_{1,2}) < 0$ and the steady motion at σ_m is stable.

Thus, only in Case IIb, when both stiffness and damping-in-pitch derivatives are positive, is the steady motion at angle of attack σ_m stable to infinitesimal disturbances. In fact, stability theory[3] can be used to show that stability of the steady motion in this case is assured only if the disturbance is sufficiently small.

It can happen that even if stability to infinitesimal disturbances is assured, a disturbance of sufficient magnitude will cause the steady motion to become unstable. An example of these possibilities is provided in Fig. 1 for a flat-plate airfoil flying in air ($\gamma = 1.4$) at supersonic speed $M_\infty = 2$. For a center of gravity at the leading edge, that is, $h = 0$, via linear theory[7,8] one predicts stability for $\sigma_m < \sigma_{cr} = 16.25$ deg and instability for $\sigma_m > \sigma_{cr}$. At $\sigma_m = 15.75$ deg $\equiv \sigma_1 < \sigma_{cr}$, the damping-in-pitch coefficient is small and Eqs. (20) apply with $F(\xi,\dot{\xi},\sigma_m)$ calculated in the way explained in Sec. IIC. Figure 1a shows the time-history of the displacement angle ξ after it is released from an initial departure $\xi(0) = 2.5$ deg from $\sigma_m = \sigma_1$. It is evident that ξ is decaying toward zero as $\tau \to \infty$, indicating that the steady flight condition $\sigma_m = \sigma_1$ is stable. In contrast, if the initial departure from $\sigma_m = \sigma_1$ is $\xi(0) = 4$ deg, the time-history in Fig. 1b shows that ξ does *not* decay toward zero as $\tau \to \infty$, indicating that the steady motion $\sigma_m = \sigma_1$ is unstable to the larger disturbance.

In all other cases, that is, when either the stiffness derivative or the damping derivative is negative, the steady motion at angle of attack σ_m is unstable, no matter how small

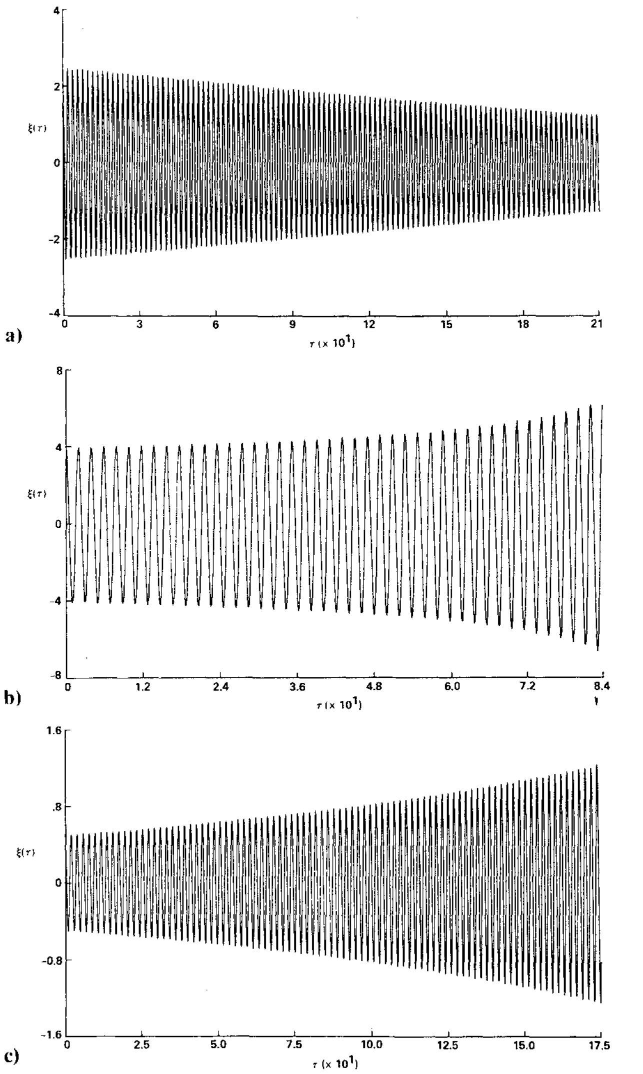


Fig. 1 Time history $\xi(\tau)$ of a flat-plate airfoil pitching about its leading edge in a supersonic freestream: $M_\infty = 2.0$, $\gamma = 1.4$, $\kappa = 1$, neutral damping at $\sigma_{cr} = 16.25$ deg. a) $\sigma_m = 15.75$ deg, $\xi(0) = 2.5$ deg; $\dot{\xi}(0) = 0$; b) $\sigma_m = 15.75$ deg, $\xi(0) = 4$ deg, $\dot{\xi}(0) = 0$; c) $\sigma_m = 16.75$ deg, $\xi(0) = 0.5$ deg, $\dot{\xi}(0) = 0$.

the disturbance. As an example, the time-history of the displacement angle ξ of the same flat-plate airfoil at $M_\infty = 2$, now with $\sigma_m = \sigma_2 = \sigma_{cr} + 0.5$ deg and $\xi(0) = 0.5$ deg, is shown in Fig. 1c. The growth of $\xi(\tau)$ as $\tau \to \infty$ clearly indicates the instability of the steady flight condition at $\sigma_m = \sigma_2 > \sigma_{cr}$.

In all cases in which the linearized theory predicts growth of the disturbance amplitude, the growth predicted is of exponential form and hence must cease to be valid after some finite time when the amplitude is no longer small. Thus, what eventually happens to a motion for which linearized stability theory predicts an initial growth of disturbances cannot be determined from the linearized theory itself. Instead, the full nonlinear inertial equations of motion, or a suitable approximation of them, such as Eqs. (20), must be adopted to determine the ultimate state of the motion. Of particular interest is the dynamic stability boundary $\sigma_m = \sigma_{cr}$, where $S(\sigma_{cr}) > 0$, $D(\sigma_{cr}) = 0$. The stability characteristics near this boundary will be studied in the next section.

IV. Nonlinear Stability Theory

At the dynamic stability boundary $\sigma = \sigma_{cr}$ we have $S(\sigma_{cr}) > 0$ and $D(\sigma_{cr}) = 0$. Hence,

$$\lambda_{\substack{1\\2}}(\sigma_{cr}) = \pm i\sqrt{\kappa S(\sigma_{cr})} \equiv \pm i\omega_0 \tag{25}$$

The existence of purely imaginary eigenvalues of the matrix A at $\sigma_m = \sigma_{cr}$ is the characteristic sign of a Hopf bifurcation,[3,13] signaling a changeover from stable steady motion to periodic motion. On crossing $\sigma_m = \sigma_{cr}$, the steady motion that had been stable for $\sigma_m < \sigma_{cr}$ will become unstable to disturbances, resulting (after a transient motion has died away) in the existence of a new motion, which (if it is stable) will be periodic. In the vicinity of $\sigma_m = \sigma_{cr}$, the circular frequency of the periodic motion will be nearly equal to ω_0. We call the new solution of the equations of motion a bifurcation solution; in this section we shall determine its character and a criterion for its stability.

For σ_m slightly larger than σ_{cr}, the eigenvalues of the matrix A are

$$\lambda_{\substack{1\\2}} = -\tfrac{1}{2}\kappa D(\sigma_m) \pm i\Omega(\sigma_m) \tag{26}$$

where

$$\Omega(\sigma_m) = \sqrt{\kappa S(\sigma_m) - \frac{\kappa^2 D^2(\sigma_m)}{4}} \tag{27}$$

We shall assume that

$$D'(\sigma_{cr}) < 0 \tag{28}$$

which is the usual case in applications.[6] [The case $D'(\sigma_{cr}) > 0$ can be treated in exactly the same way.] The normalized eigenvector $\zeta(\sigma_m)$ associated with the eigenvalue $\lambda(\sigma_m)$ is

$$\zeta(\sigma_m) = \begin{pmatrix} \zeta_1(\sigma_m) \\ \zeta_2(\sigma_m) \end{pmatrix} = \frac{1-i}{2\sqrt{\Omega(\sigma_m)}} \begin{pmatrix} 1 \\ \lambda(\sigma_m) \end{pmatrix} \tag{29}$$

whereas the adjoint eigenvector $\zeta^*(\sigma_m)$ with eigenvalue $\bar{\lambda}(\sigma_m)$, which is the complex conjugate of $\lambda(\sigma_m)$, is

$$\zeta^*(\sigma_m) = \begin{pmatrix} \zeta_1^*(\sigma_m) \\ \zeta_2^*(\sigma_m) \end{pmatrix} = \frac{1+i}{2\sqrt{\Omega(\sigma_m)}} \begin{pmatrix} \bar{\lambda}(\sigma_m) + \kappa D(\sigma_m) \\ 1 \end{pmatrix} \tag{30}$$

A. Hopf Bifurcation

The bifurcation solution $u(\tau, \sigma_m)$ may be written as

$$u = a(\tau)\zeta + \bar{a}(\tau)\bar{\zeta} \tag{31}$$

where the complex amplitude function $a(\tau, \sigma_m)$ is determined from Eq. (21) as follows. From Eq. (31)

$$\dot{u}_i = \dot{a}\zeta_i + \dot{\bar{a}}\bar{\zeta}_i \tag{32}$$

Substituting Eq. (32) into Eq. (21), multiplying by $\bar{\zeta}_i^*$, and summing over i yields

$$\begin{aligned} \dot{a} &= \mathfrak{F}(a, \sigma_m) \\ &= \lambda(\sigma_m)a + \alpha_2(\sigma_m)a^2 + 2\beta_2(\sigma_m)\,|a|^2 \\ &\quad + \gamma_2(\sigma_m)\bar{a}^2 + \alpha_3(\sigma_m)a^3 + \beta_{31}(\sigma_m)a\,|a|^2 \\ &\quad + \beta_{32}(\sigma_m)\bar{a}\,|a|^2 + \gamma_3(\sigma_m)\bar{a}^3 + 0(\,|a|^4) \end{aligned} \tag{33}$$

where

$$\alpha_2(\sigma_m) = B_{ijk}(\sigma_m)\zeta_j\zeta_k\bar{\zeta}_i^* \tag{34}$$

$$\beta_2(\sigma_m) = B_{ijk}(\sigma_m)\zeta_j\bar{\zeta}_k\bar{\zeta}_i^* \tag{35}$$

$$\gamma_2(\sigma_m) = B_{ijk}(\sigma_m)\bar{\zeta}_j\bar{\zeta}_k\bar{\zeta}_i^* \tag{36}$$

$$\alpha_3(\sigma_m) = C_{ijk\ell}(\sigma_m)\zeta_j\zeta_k\zeta_\ell\bar{\zeta}_i^* \tag{37}$$

$$\beta_{31}(\sigma_m) = 3C_{ijk\ell}(\sigma_m)\zeta_j\zeta_k\bar{\zeta}_\ell\bar{\zeta}_i^* \tag{38}$$

$$\beta_{32}(\sigma_m) = 3C_{ijk\ell}(\sigma_m)\zeta_j\bar{\zeta}_k\bar{\zeta}_\ell\bar{\zeta}_i^* \tag{39}$$

$$\gamma_3(\sigma_m) = C_{ijk\ell}(\sigma_m)\bar{\zeta}_j\bar{\zeta}_k\bar{\zeta}_\ell\bar{\zeta}_i^* \tag{40}$$

To solve Eq. (33), we follow Iooss and Joseph (Ref. 3, p. 125), letting

$$\text{at} \quad a(\tau,\sigma_m) = b(s,\epsilon)$$

$$s = \omega(\epsilon)\tau, \qquad \omega(0) = \omega_0 = \sqrt{\kappa S(\sigma_{cr})}$$

$$\sigma_m = \sigma_m(\epsilon), \qquad \sigma_m(0) = \sigma_{cr} \tag{41}$$

where ϵ is an amplitude parameter defined by

$$\epsilon = \frac{1}{2\pi}\int_0^{2\pi} e^{-is} b(s,\epsilon)\,\mathrm{d}s \equiv [b] \tag{42}$$

which is a measure of the departure from the dynamic stability boundary. With the transformation of Eqs. (41), Eq. (33) is equivalent to the following system:

$$\omega\frac{\partial b}{\partial s} = \mathfrak{F}(b,\sigma_m(\epsilon)), \qquad \epsilon = [b] \tag{43}$$

Again following Iooss and Joseph,[3] we look for a series solution of Eqs. (43) in the following form:

$$b(s,\epsilon) = \sum_{n=1}^{\infty} b_n(s)\epsilon^n$$

$$\sigma_m(\epsilon) = \sigma_{cr} + \sum_{n=1}^{\infty}\sigma_{mn}\epsilon^n$$

$$\omega(\epsilon) = \omega_0 + \sum_{n=1}^{\infty}\omega_n\epsilon^n \tag{44}$$

Substituting Eqs. (44) into Eqs. (43) and equating like terms in ϵ in the resulting equations yields a sequence of equations for the successive determinations of $(b_n(s),\sigma_{mn},\omega_n)$, $n=1,2,3,\ldots$ [A subscript $(\ )_0$ of a function of σ_m denotes its value when the argument is σ_{cr}; e.g., $\lambda_0 = \lambda(\sigma_{cr})$, $\lambda_0' = \mathrm{d}\lambda(\sigma_{cr})/\mathrm{d}\sigma_m$, $\beta_{20} = \beta_2(\sigma_{cr})$, $\beta_{310} = \beta_{31}(\sigma_{cr})$.] To $0(\epsilon)$,

$$\omega_0\frac{\partial b_1}{\partial s} = \lambda_0 b_1 = i\omega_0 b_1, \qquad \int_0^{2\pi} e^{-is} b_1(s)\,\mathrm{d}s = 2\pi \tag{45}$$

which has a unique solution

$$b_1 = e^{is} \tag{46}$$

a periodic function with period 2π in s. Collecting terms of $0(\epsilon^2)$ yields

$$\omega_0\left(\frac{\partial b_2}{\partial s} - ib_2\right) = (-i\omega_1 + \lambda_0'\sigma_{m1})e^{is} + \alpha_{20}e^{2is}$$

$$+2\beta_{20} + \gamma_{20}e^{-2is}, \qquad \int_0^{2\pi} e^{-is} b_2(s)\,\mathrm{d}s = 0 \tag{47}$$

The function b_2 will be periodic in s if and only if the coefficient of the term e^{is} on the right-hand side of the first of Eqs. (47) vanishes, which requires

$$\lambda_0'\sigma_{m1} - i\omega_1 = 0 \tag{48}$$

Since

$$Re(\lambda_0') = -\frac{\kappa}{2}D'(\sigma_{cr}) > 0 \tag{49}$$

we have

$$\sigma_{m1} = \omega_1 = 0 \tag{50}$$

Equations (47) may then be simplified to yield

$$b_2(s) = \frac{1}{i\omega_0}\left[\alpha_{20}e^{2is} - 2\beta_{20} - \frac{\gamma_{20}}{3}e^{-2is}\right] \tag{51}$$

Proceeding to the next order in ϵ gives

$$\omega_0\left(\frac{\partial b_3}{\partial s} - ib_3\right) = \left[\lambda_0'\sigma_{m2} - i\omega_2 + \frac{2}{i\omega_0}\right.$$
$$\left.\times\left(-\alpha_{20}\beta_{20} + \frac{|\gamma_{20}|^2}{3} + 2|\beta_{20}|^2\right) + \beta_{310}\right]e^{is}$$
$$+\frac{1}{i\omega_0}\left[\left(2\alpha_{20}^2 + \frac{2}{3}\beta_{20}\bar{\gamma}_{20} + i\omega_0\alpha_{30}\right)e^{3is}\right.$$
$$-\left(\frac{2}{3}\beta_{20}\gamma_{20} + 2\bar{\alpha}_{20}\gamma_{20} + i\omega_0\gamma_{30}\right)e^{-3is}$$
$$\left.-\left(\frac{2}{3}\alpha_{20}\gamma_{20} - 4\bar{\beta}_{20}\gamma_{20} + 4\beta_{20}^2 + 2\bar{\alpha}_{20}\beta_{20} - i\omega_0\beta_{320}\right)e^{-is}\right],$$
$$\int_0^{2\pi} e^{-is} b_3(s)\,\mathrm{d}s = 0 \tag{52}$$

Again, the factor multiplying e^{is} on the right-hand side of the first of Eqs. (52) must vanish in order that the solution for $b_3(s)$ be periodic. Using the condition Eq. (49) yields

$$\sigma_{m2} = \frac{1}{\kappa D'(\sigma_{cr})}\left[\frac{-4}{\omega_0}Im(\alpha_{20}\beta_{20}) + 2Re(\beta_{310})\right] \tag{53}$$

and

$$\omega_2 = Im(\lambda_0')\sigma_{m2} + Im(\beta_{310})$$
$$+\frac{1}{\omega_0}\left[2Re(\alpha_{20}\beta_{20}) - 4|\beta_{20}|^2 - \frac{2}{3}|\gamma_{20}|^2\right] \tag{54}$$

The solution for $b_3(s)$ is

$$b_3(s) = -\frac{1}{\omega_0^2}\left[\left(\alpha_{20}^2 + \frac{\beta_{20}\bar{\gamma}_{20}}{3} + \frac{1}{2}i\omega_0\alpha_{30}\right)e^{3is}\right.$$
$$+\left(\frac{\alpha_{20}\gamma_{20}}{3} - 2\bar{\beta}_{20}\gamma_{20} + 2\beta_{20}^2 + \bar{\alpha}_{20}\beta_{20}\right.$$
$$\left.\left.-\frac{1}{2}i\omega_0\beta_{320}\right)e^{-is} + \frac{1}{2}\left(\bar{\alpha}_{20}\gamma_{20} + \frac{\beta_{20}\gamma_{20}}{3} - \frac{1}{2}i\omega_0\gamma_{30}\right)e^{-3is}\right] \tag{55}$$

The solution procedure given above can be continued to higher orders in ϵ. In particular, a recursion formula for determining $b_n(s)$ is derived in Ref. 14, which shows that

$$\sigma_{m(2n-1)} = \omega_{2n-1} = 0, \qquad n = 1,2,\ldots \tag{56}$$

rendering $\omega(\epsilon)$ and $\sigma_m(\epsilon)$ even functions.

To sum up, the periodic bifurcation solution to $0(\epsilon^4)$ is given by

$$u = a\zeta + \bar{a}\bar{\zeta} \tag{57a}$$

with

$$a = \epsilon b_1(s) + \epsilon^2 b_2(s) + \epsilon^3 b_3(s) + 0(\epsilon^4) \tag{57b}$$

$$s = [\omega_0 + \epsilon^2\omega_2 + 0(\epsilon^4)]\tau \tag{57c}$$

$$\sigma_m = \sigma_{cr} + \epsilon^2\sigma_{m2} + 0(\epsilon^4) \tag{57d}$$

The solution is periodic in τ with circular frequency equal to $\omega_0 + \epsilon^2\omega_2 + 0(\epsilon^4)$.

B. Stability of the Periodic Bifurcation Solution

According to Floquet theory,[3] the stability of the periodic bifurcation solution, Eq. (57), is determined by the sign of an index μ. To $0(\epsilon^4)$, μ has the form

$$\mu = \kappa D'(\sigma_{cr})\sigma_{m2}\epsilon^2 + 0(\epsilon^4) \tag{58}$$

and the periodic bifurcation solution is stable if $\mu<0$ and unstable if $\mu>0$. Since we have assumed $D'(\sigma_{cr})<0$, stability thus depends on the sign of σ_{m2}; $\sigma_{m2}>0$ denoting stability, $\sigma_{m2}<0$ denoting instability. It remains to cast σ_{m2} in more recognizable terms. Using Eq. (53), we have

$$\mu = \left[\frac{-4}{\omega_0} Im(\alpha_{20}\beta_{20}) + 2Re(\beta_{310})\right]\epsilon^2 + 0(\epsilon^4) \tag{59}$$

From Eqs. (34)-(40), and using Eqs. (22), (23), (29), and (30), we find

$$\alpha_{20} = \alpha_2(\sigma_{cr}) = B_{ijk}(\sigma_{cr})\zeta_j(\sigma_{cr})\zeta_k(\sigma_{cr})\zeta_i^*(\sigma_{cr})$$

$$= -\left(\frac{1+i}{4\omega_0^{3/2}}\right)[B_{211} - \omega_0^2 B_{222} + 2i\omega_0 B_{212}]_{\sigma_m=\sigma_{cr}} \tag{60}$$

Similarly,

$$\beta_{20} = \left(\frac{1-i}{4\omega_0^{3/2}}\right)[B_{211} + \omega_0^2 B_{222}]_{\sigma_m=\sigma_{cr}}$$

Hence,

$$\frac{-4}{\omega_0} Im(\alpha_{20}\beta_{20}) = \frac{1}{\omega_0^3}[B_{211}B_{221} + \omega_0^2 B_{212}B_{222}] \tag{61}$$

On the other hand, using Eq. (22) and the symmetry properties Eq. (23) yields

$$2\beta_{310} = 6C_{2jk\ell}(\sigma_{cr})\zeta_j\zeta_k\zeta_\ell\bar{\zeta}_2^*|_{\sigma_m=\sigma_{cr}}$$

$$= C_{2jk}(\sigma_{cr})\bar{\zeta}_2^*(\sigma_{cr})[\zeta_\ell(\zeta_j\zeta_k + \bar{\zeta}_j\bar{\zeta}_k)$$

$$+ \zeta_j(\bar{\zeta}_k\zeta_\ell + \zeta_k\bar{\zeta}_\ell) + \zeta_k(\zeta_j\bar{\zeta}_l + \bar{\zeta}_j\zeta_\ell)]_{\sigma_m=\sigma_{cr}}$$

or

$$\beta_{310} = C_{2jk\ell}(\sigma_{cr})\bar{\zeta}_2^*(\sigma_{cr})[R_{jk}\zeta_\ell + R_{k\ell}\zeta_j + R_{\ell j}\zeta_k]_{\sigma_m=\sigma_{cr}}$$

where

$$R_{jk} = Re(\bar{\zeta}_j\zeta_k)|_{\sigma_m=\sigma_{cr}}$$

With the aid of Eqs. (29) and (30),

$$R_{11} = \frac{1}{2\omega_0}, \quad R_{12} = R_{21} = 0, \quad R_{22} = \frac{\omega_0}{2}$$

and hence

$$2Re(\beta_{310}) = \frac{3}{2\omega_0^2}[C_{2112} + \omega_0^2 C_{2222}] \tag{62}$$

Substituting Eqs. (61) and (62) into (59) yields the following expression for the index μ:

$$\mu = \frac{\epsilon^2}{2\omega_0^3}[2(B_{211}B_{212} + \omega_0^2 B_{212}B_{222})$$

$$+ 3\omega_0^2(C_{2112} + \omega_0^2 C_{2222})]_{\sigma_m=\sigma_{cr}} + 0(\epsilon^4) \tag{63}$$

or, alternatively, after identifying the coefficients in Eq. (63) with the derivatives of $F(u_1,u_2,\sigma_m)$ in the Taylor series in Eqs. (21) and (22), we have

$$\mu = \frac{\epsilon^2}{4\omega_0^3}\left[\left(\frac{\partial^2 F}{\partial u_1^2} + \omega_0^2\frac{\partial^2 F}{\partial u_2^2}\right)\frac{\partial^2 F}{\partial u_1 \partial u_2} + \omega_0^2\left(\frac{\partial^3 F}{\partial u_1^2 \partial u_2} + \omega_0^2\frac{\partial^3 F}{\partial u_2^3}\right)\right]_{\substack{u=0\\ \sigma_m=\sigma_{cr}}} \tag{64}$$

in terms of $F(u_1,u_2,\sigma_m)$. From Eq. (19) we see that the function F is directly related to the pitching-moment coefficient $C_m(\xi,\dot{\xi},\sigma_m)$ acting on the aircraft which is performing a finite-amplitude pitching oscillation ξ around a mean angle of attack σ_m. Equation (64) demonstrates that the stability of the periodic motion near the dynamic stability boundary σ_{cr} is determined by the behavior of the aerodynamic response $C_m(\xi,\dot{\xi},\sigma_m)$ in that vicinity.

With the assumption of slow oscillations under which the form of Eq. (19) was derived [terms of $0(\dot{\xi}^2,\ddot{\xi})$ neglected], we may substitute Eq. (19) into Eq. (64) to get

$$\mu = \frac{\epsilon^2}{4\omega_0^3}\left[\kappa^2\left(\frac{\partial^2 f}{\partial \xi^2}\right)\left(-\frac{\partial g}{\partial \xi}\right) + \kappa\omega_0^2\left(\frac{\partial^2 g}{\partial \xi^2}\right)\right]_{\substack{\xi=0\\ \sigma_m=\sigma_{cr}}} \tag{65}$$

From Eq. (18) and the structure of the functions f and g, we write

$$\left.\frac{\partial f}{\partial \xi}\right|_{\xi=0} = -S(\sigma_m) \qquad \left.\frac{\partial^2 f}{\partial \xi^2}\right|_{\xi=0} = -S'(\sigma_m)$$

$$g|_{\xi=0} = -D(\sigma_m), \qquad \left.\frac{\partial g}{\partial \xi}\right|_{\xi=0} = -D'(\sigma_m)$$

$$\left.\frac{\partial^2 g}{\partial \xi^2}\right|_{\xi=0} = -D''(\sigma_m) \tag{66}$$

By using Eqs. (26) and (66) in Eq. (65), we arrive at the simple form [with $S(\sigma_{cr})>0$]

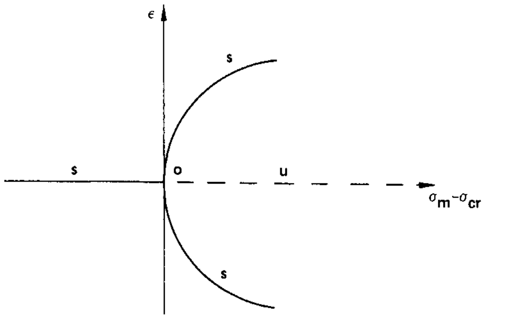
$$\mu = \frac{-\epsilon^2(\kappa S)^{1/2}}{4}\frac{\mathrm{d}}{\mathrm{d}\sigma_m}\left(\frac{D'}{S}\right)\Big|_{\sigma_m=\sigma_{cr}} \tag{67}$$

(The simplicity of this result suggests that it might be possible to derive it by a less formal method. This is indeed the case: We have verified the result by a physical approach familiar to workers in the field of nonlinear mechanics.) Equation (67) reveals that the sign of μ, and thus the stability of the bifurcation solution, is independent of the scalar and inertial properties of the aircraft. Rather, stability depends only on whether the aerodynamic property (D'/S) is increasing or decreasing on crossing the dynamic stability boundary $\sigma_m=\sigma_{cr}$. The two possibilities are well illustrated in the form of bifurcation diagrams, as shown in Figs. 2a and 2b.

In a bifurcation diagram, the abscissa represents the parameter that is being varied; in our case, the mean angle of attack σ_m. The ordinate is a parameter characteristic of the bifurcation solution alone; in our case it is ϵ [Eq. (42)], a measure of the amplitude of the periodic bifurcation solution. Stable solutions are indicated by solid lines, unstable solutions by dashed lines. Thus, over the range of mean angle of attack $\sigma_m<\sigma_{cr}$, where the steady-state motion is stable, ϵ is zero and the stable steady motion is represented along the abscissa by a solid line. The steady motion becomes unstable for all values of $\sigma_m>\sigma_{cr}$ as the dashed line along the abscissa indicates. Periodic solutions bifurcate from $\sigma_m=\sigma_{cr}$ either supercritically or subcritically.

When

$$\frac{\mathrm{d}}{\mathrm{d}\sigma_m}\left(\frac{D'}{S}\right)\Big|_{\sigma_m=\sigma_{cr}} > 0$$



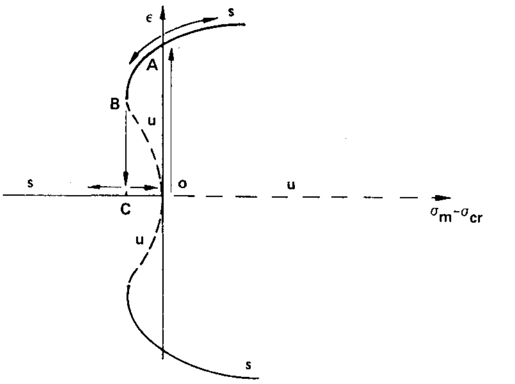


Fig. 2 Typical forms of bifurcation diagrams near the dynamic stability boundary σ_{cr} where $D(\sigma_{cr})=0$. a) Supercritical,

$$\frac{d}{d\sigma_m}\left[\frac{D'(\sigma_m)}{S(\sigma_m)}\right]_{\sigma_m=\sigma_{cr}}>0;\ \text{b) subcritical,}\ \frac{d}{d\sigma_m}\left[\frac{D'(\sigma_m)}{S(\sigma_m)}\right]_{\sigma_m=\sigma_{cr}}<0.$$

(implying $\sigma_{m2}>0$), the bifurcation is called supercritical and its characteristic form is shown in Fig. 2a. Stable periodic solutions (solid curves in Fig. 2a) exist for values of $\sigma_m-\sigma_{cr}>0$. The amplitude of the periodic solution at a given value of $\sigma_m-\sigma_{cr}$ is proportional to ϵ and hence is vanishingly small when $\sigma_m-\sigma_{cr}$ is small, varying essentially as $(\sigma_m-\sigma_{cr})^{1/2}$.

When

$$\frac{d}{d\sigma_m}\left(\frac{D'}{S}\right)\bigg|_{\sigma_m=\sigma_{cr}}<0$$

(implying $\sigma_{m2}<0$), the bifurcation is called subcritical and its characteristic form is shown in Fig. 2b. Periodic solutions exist for values of $\sigma_m-\sigma_{cr}<0$, but they are unstable (dashed curve in Fig. 2b). Whether or not stable periodic solutions exist for $\sigma_m>\sigma_{cr}$ depends predominantly on the behavior of the damping-in-pitch derivative $D(\sigma_m)$ for $\sigma_m>\sigma_{cr}$. If no such stable periodic solutions exist for $\sigma_m>\sigma_{cr}$, then when the mean angle of attack σ_m is increased beyond σ_{cr}, the aircraft may undergo an aperiodic motion whose departure from the steady motion at $\sigma_m=\sigma_{cr}$ is potentially large.

In the more likely event that stable periodic solutions do exist for $\sigma_m>\sigma_{cr}$ (an example is given later), their amplitudes must be finite, and not infinitesimally small, even for small positive values of $\sigma_m-\sigma_{cr}$. It is likely that this branch of stable periodic solutions will join that of the unstable branch in the way illustrated in Fig. 2b. In this event, the form of the bifurcation curve for values of $\sigma_m<\sigma_{cr}$ helps explain the situation discussed earlier in connection with Figs. 1b and 1c, where it was noted that the steady-state motion could be stable to sufficiently small disturbances but become unstable to larger disturbances. Thus, Fig. 2b suggests that for $\sigma_m<\sigma_{cr}$, as long as disturbances are of small enough amplitude to lie below those of the unstable branch of periodic solutions (curve *OB* in Fig. 2b), then they will die out with time and the steady motion will remain stable. However, disturbances with amplitudes sufficiently larger than those of the unstable branch may actually grow up to the ultimate motion as $\tau\to\infty$, which will be that of the stable branch of period solutions (curve *BA* in Fig. 2b). Finally, we note that if the motion does attain the stable branch of periodic soluions (say, for $\sigma_m<\sigma_{cr}$) then hysteresis effects will manifest themselves with further changes in σ_m. When σ_m is increased beyond σ_{cr}, the motion will continue to be periodic with finite amplitude (point *A* in Fig. 2b). If σ_m is now decreased below σ_{cr}, the periodic motion will persist, even at values of σ_m where previously there had been steady motion when σ_m was being increased. Not until σ_m is diminished beyond a certain point (point *B* in Fig. 2b) will the motion return to the steady-state condition (point *C* in Fig. 2b) that had been experienced when σ_m was increasing.

In order to further explore the implications of Eq. (67), we invoke some approximate relationships between the damping-in-pitch derivative $D(\sigma_m)$ and the stiffness derivative $S(\sigma_m)$. Tobak and Schiff have argued (see, in particular Ref. 15) that, to good accuracy, a linear relationship should exist between D and S at any value of σ_m; that is, $D(\sigma_m)\approx a-bS(\sigma_m)$, with $b>0$. In addition, we require that D vanish at $\sigma_m=\sigma_{cr}$. The two requirements yield the form

$$D(\sigma_m)=b[S(\sigma_{cr})-S(\sigma_m)]\qquad b>0 \tag{68}$$

(The validity of the approximate relation, Eq. (68), has been verified in the present study (via comparison with exact results in Ref. 7) for use with oscillating flat-plate airfoils in supersonic/hypersonic flow and in Ref. 11 for use with oscillating flaps in transonic flow.) Replacing D in Eq. (67) by Eq. (68) casts the criterion solely in terms of S:

$$\mu=\frac{\epsilon^2(\kappa S)^{1/2}b}{4}\frac{d^2}{d\sigma_m^2}\left(\ln S(\sigma_m)\right)\big|_{\sigma_m=\sigma_{cr}} \tag{69}$$

Thus, near $\sigma_m=\sigma_{cr}$, if the stiffness derivative $S(\sigma_m)$ increases with σ_m slower/faster than exponential, the periodic bifurcation solution is stable (supercritical)/unstable (subcritical). Cast in terms of D instead of S, the criterion states that *decreasing D* with respect to σ_m slower/faster than exponential results in stable (supercritical)/unstable (subcritical) periodic bifurcation solutions.

Examples of $S(\sigma_m)$ and $D(\sigma_m)$ that obey Eq. (68) and exhibit the various possibilities are as follows.

Example 1:

$$S_1(\sigma_m)=S_0\exp[(k(\sigma_m-\sigma_{cr})+m(\sigma_m-\sigma_{cr})^2]$$

$$D_1(\sigma_m)=bS_0\left[1-\exp[k(\sigma_m-\sigma_{cr})+m(\sigma_m-\sigma_{cr})^2]\right] \tag{70}$$

Example 2:

$$S_2(\sigma_m)=S_0\exp[k(\sigma_m-\sigma_{cr})+m(\sigma_m-\sigma_{cr})^2]$$

$$D_2(\sigma_m)=bS_0\left[1-\exp[k(\sigma_m-\sigma_{cr})+m(\sigma_m-\sigma_{cr})^2-n(\sigma_m-\sigma_{cr})^4]\right] \tag{71}$$

where S_0, k, and n are positive constants. According to Eq. (69), $m>0$ corresponds to unstable periodic bifurcation solutions (subcritical bifurcation), and $m<0$ corresponds to stable periodic bifurcation solutions (supercritical bifurcation). In the case of subcritical bifurcation ($m>0$) in Example 1, the damping-in-pitch derivative $D_1(\sigma_m)$ continues to decrease to very large negative values as $(\sigma_m-\sigma_{cr})$ increases, which makes it very unlikely that the system Eq. (20) will have a stable periodic motion as a solution for $\sigma_m>\sigma_{cr}$. On the other hand, in Example 2 the damping-in-pitch derivative $D_2(\sigma_m)$ becomes positive for sufficiently large $|\sigma_m-\sigma_{cr}|$ so that a stable period motion may be a possible solution for $\sigma_m>\sigma_{cr}$, $m>0$. Indeed, S_2 and D_2 in Example 2 fulfill all of the conditions set by a theorem of

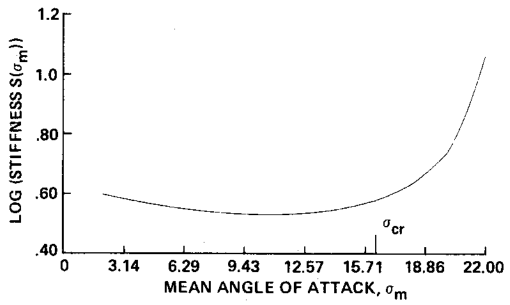


Fig. 3 Stiffness derivative $S(\sigma_m)$ vs mean angle of attack σ_m for a flat-plate airfoil in a supersonic freestream: $M_\infty = 2.0$, $h = 0$, $\gamma = 1.4$, $\kappa = 1$.

Table 1 Values of stability criterion $\mu(M_\infty, h)$ for flat-plate airfoil $\kappa = \epsilon^2 = 1$, $\gamma = 1.4$; $\mu > 0$ subcritical bifurcation; $\mu < 0$ supercritical bifurcation

M_∞ h	0	0.1	0.2	0.3	0.4
1.5	36.5	19.0	–	–	–
1.6	39.6	23.4	13.6	8.8	10.7
1.7	40.0	24.8	15.5	11.0	12.8
1.8	39.7	25.3	16.3	12.0	13.7
1.9	39.4	25.5	16.8	12.6	14.3
2.0	39.5	25.8	17.2	13.0	14.7
M_∞ h	0	0.1	0.2	0.3	0.4
2	39.5	25.8	17.2	13.0	14.7
3	57.2	38.6	26.8	20.9	23.3
4	84.7	58.4	41.4	32.8	36.4
5	107.3	74.8	53.7	42.9	47.5
6	123.6	86.7	62.6	50.3	55.6
M_∞ h	0.25	0.26	0.27	0.28	0.29
1.6	10.6	10.1	9.7	9.3	9.0
1.7	12.6	12.2	11.8	11.5	11.2
1.8	13.6	13.2	12.8	12.5	12.2
1.9	14.1	13.7	13.4	13.0	12.8
2.0	14.5	14.1	13.8	13.5	13.2
0.30	0.31	0.32	0.33	0.34	0.35
8.8	8.6	8.5	8.4	8.4	8.5
11.0	10.8	10.7	10.6	10.6	10.7
12.0	11.8	11.7	11.6	11.7	11.7
12.5	12.3	12.3	12.2	12.2	12.3
13.0	12.8	12.7	12.6	12.7	12.7

Fillippov,[16] the satisfaction of which guarantees the existence of a stable periodic motion of the system Eq. (20) for $\sigma_m > \sigma_{cr}$. As we have noted, resulting as it does from a subcritical bifurcation, the solution will exhibit hysteresis effects with variations in σ_m below σ_{cr}.

V. Supersonic/Hypersonic Flat-Plate Airfoils

To illustrate the application of bifurcation theory in a concrete case, we consider a flat-plate airfoil in supersonic/hypersonic flow. The stiffness derivative $S(\sigma_m)$ and the damping-in-pitch derivative $D(\sigma_m)$ are known as analytic functions of σ_m up to the angle for shock detachment.[6-8] They depend on the flight Mach number M_∞, the ratio of specific heats γ (here taken to be that of air, $\gamma = 1.4$), and the (dimensionless) distance h of the center of gravity from the leading edge, here taken as a fraction of the chord length ℓ. Results are presented in Fig. 3 of log $S(\sigma_m)$ vs σ_m (with $M_\infty = 2.0$, $h = 0$) and in Table 1 of the index μ for various combinations of M_∞ and h. It is shown in Fig. 3 that $S(\sigma_m)$ increases faster than exponential near $\sigma_m = \sigma_{cr} = 15.75$ deg, and in Table 1 that μ is always positive. We conclude that whenever the flat-plate airfoil becomes dynamically unstable $[D(\sigma_{cr}) \equiv 0]$, the ensuing bifurcation will always be subcritical.

Accordingly, there are two possibilities. One, a subcritical bifurcation curve, such as that sketched in Fig. 2b, exists, in which case the airfoil motion will find stability at values of $\sigma_m > \sigma_{cr}$ in a periodic oscillation of finite amplitude. At values of $\sigma_m < \sigma_{cr}$, the steady-state motion will be stable to small disturbances, but for larger disturbances the airfoil motion will again seek the stable periodic motion. Exchanges between these two stable modes at $\sigma_m < \sigma_{cr}$ will be accompanied by hysteresis effects. Two, alternatively, a bifurcation curve, such as that sketched in Fig. 2b, does not exist, in which case a potentially large aperiodic departure from the steady-state motion may occur as σ_m exceeds σ_{cr}. In either case, on exceeding σ_{cr} the loss of stability of the steady-state motion must entail a *discrete* change to a new stable condition.

VI. Concluding Remarks

Bifurcation theory has been used to analyze the nonlinear dynamic stability characteristics of an aircraft subject to single-degree-of-freedom pitching-motion perturbations about a large mean angle of attack. Setting up the equations to which the bifurcation theory was applied required 1) determining conditions under which the inertial equations of motion and the gasdynamic equations governing the flow could be decoupled; and 2) showing how the required aerodynamic responses to finite-amplitude oscillations could be obtained from the responses to infinitesimal-amplitude oscillations.

Results of the bifurcation theory analysis revealed that when the mean angle of attack is increased past the critical point where the aerodynamic damping vanishes, new solutions describing finite-amplitude periodic motions bifurcate from the previously stable steady motion. The sign of a simple criterion, cast in terms of aeodynamic properties, determines whether the bifurcating solutions are stable (supercritical) or unstable (subcritical). For flat-plate airfoils flying at supersonic/hypersonic speed, the bifurcation is subcritical, implying either that exchanges of stability between steady and periodic motions will be accompanied by hysteresis phenomena, or that a potentially dangerous aperiodic motion may develop. In either case the loss of stability of the steady-state motion must be acompanied by a discrete change to a new stable state.

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